Comparison of spheroidal and eigenfunction-expansion trial functions for a membrane in an infinite baffle

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The on-axis far-field pressure response of a circular membrane in an infinite baffle when driven by a uniformly distributed electrostatic force is calculated using two different trial functions for the surface velocity distribution. The first is an expansion based upon a solution to the free space wave equation in oblate spheroidal coordinates, which has already been derived in a previous paper [J. Acoust. Soc. Am. **120**(5), 2460–2477 (2006)], and the second is a membrane eigenfunction expansion (or Bessel series), which is rigorously derived in this letter. Although the latter can be used as a basis for calculating a number of different radiation characteristics such as the radiation impedance or directivity, etc., only the on-axis far-field sound pressure is considered here. The results are compared and discussed. © *2008 Acoustical Society of America.*
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I. INTRODUCTION

In a previous paper,[1] the sound field of a circular membrane in an infinite baffle was calculated using a trial function for the surface velocity distribution based upon a solution to the free space wave equation in oblate spheroidal coordinates. No convergence problems were encountered because the chosen trial function has an inherent zero at the rim which matches the clamped rim boundary condition of the membrane. In a more recent paper,[2] the sound field of a shallow spherical shell was calculated, which differed from the membrane in that the surface velocity was non-zero at the rim. Initially, the authors tried using the spheroidal trial function, but encountered convergence problems because a very large number of terms were needed in the expansion in order to maintain the true non-zero velocity close to the rim where it is forced to zero by the trial function. Hence, an eigenfunction expansion based upon the solution to the wave equation of the shell itself was used instead. In this letter, this eigenfunction expansion method is retrospectively applied to the membrane and compared with the original spheroidal trial function. The authors originally obtained this result in order to verify the eigenfunction method before applying it to the shell. However, there are some differences in the derivations for the membrane and shell due to differences in their boundary conditions, as will be pointed out.

In Sec. II, both trial functions are presented in the form of summations with unknown coefficients. In Secs. III and IV, solutions to the free space and membrane wave equations respectively are given. In Sec. V, the solutions are then coupled and solved simultaneously in Sec. VI for the unknown trial function coefficients. Finally, in Sec. VII, expressions for the far-field sound pressure response are derived.

Electrostatic loudspeakers often have damping in the form of narrow sound holes in the electrodes which provide viscous losses in order to flatten what would otherwise be a highly modal response. The pressure response is calculated here without damping in order to obtain a response with strong modal features. This makes it easier to see whether the two calculation methods actually give the same results. Also, the lack of damping demonstrates more clearly the effect of coupling to the acoustic radiation load: Due to the radiation mass being considerably greater than the membrane mass, the modes are shifted downwards at low frequencies and the radiation resistance has a damping effect at high frequencies.

II. TRIAL FUNCTION

In the case of the membrane,[1] the following spheroidal trial function was used for the velocity distribution:

$$\tilde{u}_0(w_0) = \sum_{m=0}^{\infty} \tilde{A}_m \left(1 - \frac{w_0^2}{a^2}\right)^{m+(1/2)}, \tag{1}$$

where $\tilde{A}_m$ are the expansion coefficients (which have to be calculated), w_0 is the radial ordinate, and a is the radius of the membrane, as shown in Fig. 1. In the case of the shell,[2] an eigenfunction expansion was used instead, in order to guarantee convergence. A similar eigenfunction expansion can also be used as a trial function for a membrane as follows:

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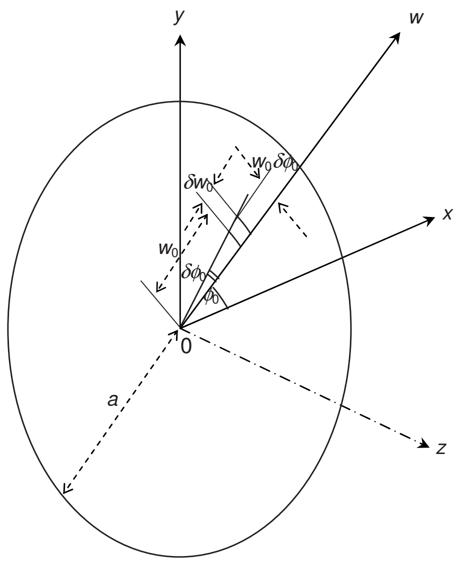

FIG. 1. Geometry of the membrane.

$$\tilde{u}_0(w_0) = \sum_{m=1}^{\infty} \tilde{A}_m J_0(\alpha_m w_0/a), \quad (2)$$

where α_m is the mth zero of $J_0(\alpha_m)$. However, this differs from the eigenfunction expansion for a shell in that a shell has bending stiffness which gives rise to a $I_0(\alpha_m w_0/a)$ term where I_0 is the Bessel function of the second kind. The shell also has a piston term which is proportional to the dome height. The uniform driving force is not expected to excite any axial modes, so only the 0th order Bessel function is included. Both of these trial functions satisfy the boundary condition of zero velocity at the rim of the membrane. The first is a solution to the wave equation for the surrounding medium to which the membrane is coupled and the second is a solution to the homogeneous wave equation for membrane itself. Only the latter is applied in this letter. In some ways, this method is slightly more direct because a set of simultaneous equations for the expansion coefficients is generated by simply equating the coefficients of $J_0(\alpha_n w/a)$ in the coupled equation. In the previous spheroidal derivation,[1] Eq. (1) was expanded as a Bessel series at one stage in order to generate the simultaneous equations. Both of these methods are fairly direct, though, and avoid the need for collocation or least squares minimization, which in turn leads to smaller errors and greater detail at high frequencies. Errors come mainly from the truncation of the expansion limits, as will be discussed.

III. SOLUTION TO THE FREE SPACE WAVE EQUATION

Using the King integral and doubling the amplitude due to the presence of the baffle, the pressure distribution is defined by

$$\tilde{p}(w,z) = 2\int_0^{2\pi}\int_0^a g(w,z|w_0,z_0) \times \frac{\partial}{\partial z_0}\tilde{p}(w_0,z_0)\big|_{z_0=0+} w_0 dw_0 d\phi_0, \quad (3)$$

where the Green's function is defined, in axisymmetric cylindrical coordinates, by

$$g(w,z|w_0,z_0) = \frac{i}{4\pi}\int_0^\infty J_0(\mu w)J_0(\mu w_0)\frac{\mu}{\sigma}e^{-i\sigma|z-z_0|}d\mu \quad (4)$$

and $\sigma=\sqrt{k^2-\mu^2}$, where k is the wave number given by $k=\omega/c=2\pi/\lambda$ and

$$\frac{\partial}{\partial z}\tilde{p}(w,z)\big|_{z=0\pm} = -ik\rho c\tilde{u}_0(w), \quad 0 \le w \le a, \quad (5)$$

where ρ is the density of air or any surrounding acoustic medium and c is the speed of sound in that medium. Inserting Eqs. (4) and (5) in Eq. (3) and integrating over the surface of the membrane while setting $z_0=0$ yields the sound pressure field as follows:

$$\tilde{p}(w,z) = ka\rho c\sum_{m=1}^{\infty}\tilde{A}_m\alpha_m J_1(\alpha_m) \times \int_0^\infty \frac{J_0(a\mu)J_0(\mu w)e^{-i\sigma z}}{(\alpha_m^2 - a^2\mu^2)\sigma}a\mu d\mu, \quad (6)$$

where the following identity[3] has been used:

$$\int_0^a J_0(\mu w_0)J_0(\alpha_m w_0/a)w_0 dw_0 = \frac{a^2}{\alpha_m^2 - a^2\mu^2}(\alpha_m J_0(a\mu)J_1(\alpha_m) - a\mu J_0(\alpha_m)J_1(a\mu)) \quad (7)$$

and taking into account that $J_0(\alpha_m)=0$. Let the power series coefficients $\tilde{A}_m$ be related to normalized dimensionless coefficients τ_m by

$$\tilde{A}_m = \tau_m\tilde{p}_I/(2\rho c), \quad (8)$$

where $\tilde{p}_I$ is the uniformly distributed driving pressure. Setting $z=0$ in Eq. (6) provides the surface pressure as follows:

$$\tilde{p}_+(w_0) = ka^2\frac{\tilde{p}_I}{2}\sum_{m=1}^{\infty}\tau_m\alpha_m J_1(\alpha_m)\int_0^\infty \frac{J_0(a\mu)J_0(\mu w_0)}{(\alpha_m^2 - a^2\mu^2)\sigma}\mu d\mu. \quad (9)$$

IV. SOLUTION TO THE MEMBRANE WAVE EQUATION

The solution[1] to the membrane wave equation $T(\nabla^2 - k_D^2)\tilde{\eta}(w) = \tilde{p}_+(w) - \tilde{p}_-(w) - \tilde{p}_I$, subject to the boundary conditions $\tilde{\eta}(a)=0$ and $\tilde{p}_+(w)=-\tilde{p}_-(w)$, is given by

$$\tilde{\eta}(w) = \frac{1}{T}\int_0^{2\pi}\int_0^a (2\tilde{p}_+(w_0) - \tilde{p}_I)G(w|w_0)w_0 dw_0 d\phi_0, \quad (10)$$

where $\nabla^2 = \partial^2/(\partial w^2) + w^{-1}\partial/(\partial w)$, $\tilde{\eta}(w)$ is the membrane deflection, T is the tension and k_D is the wave number of the membrane defined by $k_D = \omega\sqrt{\rho_D h/T}$, where ρ_D is the density of the membrane and h is its thickness. The Green's function[4] for the membrane can be written suppressing the axial term in ϕ and ϕ_0 as follows:

$$G(w|w_0) = \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{J_0(\alpha_n w/a)J_0(\alpha_n w_0/a)}{J_1^2(\alpha_n)(\alpha_n^2 - k_D^2 a^2)}, \quad 0 \le w \le a. \tag{11}$$

V. FORMULATION OF THE COUPLED PROBLEM

Substituting Eqs. (9) and (11) in Eq. (10) and equating the deflection with that given in Eq. (2) (where $\tilde{\eta}(w) = -i\tilde{u}_0(w)/kc$) leads to the following coupled equation (after integrating over the surface):

$$\frac{1}{\alpha_n J_1(\alpha_n)} = -i\tau_n \frac{\alpha_n^2 - k_D'^2 a^2}{4ka^2\rho c^2} T + ka\sum_{m=1}^{\infty} \tau_m \int_0^{\infty} \frac{\alpha_m \alpha_n J_0^2(a\mu) a\mu}{(\alpha_m^2 - a^2\mu^2)(\alpha_n^2 - a^2\mu^2)\sigma} d\mu, \tag{12}$$

which is obtained by equating the coefficients of $J_0(\alpha_n w/a)$ and again using the identity of Eq. (7) together with[5] $\int_0^a J_0(\alpha_n w_0/a) w_0 dw_0 = a^2 J_1(\alpha_n)/\alpha_n$.

VI. CALCULATION OF THE POWER SERIES COEFFICIENTS

A. Final set of simultaneous equations

From Eq. (12), the following set of M simultaneous equations in τ_m can be written

$$\sum_{m=1}^{M} {}_m\mathbf{\Psi}_n(k_D'a, ka)\tau_m = 1, \quad n = 1, 2, \ldots, M, \tag{13}$$

where ${}_m\mathbf{\Psi}_n$ is an element of the mth column an nth row of the $M \times M$ matrix given by

$${}_m\mathbf{\Psi}_n(k_D'a, ka) = \alpha_n J_1(\alpha_m)\left(-\frac{ika(\alpha_n^2 - k_D'^2 a^2)}{2\alpha^2(ka)}\delta_{mn} + \alpha_m \alpha_n I(k,m,n)\right), \tag{14}$$

where δ_{mn} is the Kronecker delta function and the infinite series limit has been truncated to order M. The dimensionless parameter α is the fluid-loading factor given by $\alpha(ka) = kac\sqrt{2a\rho/T}$. The integral $I(k,m,n)$ is defined by $I(k,m,n) = I_F(k,m,n) + iI_I(k,m,n)$, where

$$I_F(k,m,n) = ka\int_0^k \frac{J_0^2(a\mu)a\mu}{(\alpha_m^2 - a^2\mu^2)(\alpha_n^2 - a^2\mu^2)\sqrt{k^2 - \mu^2}} d\mu, \tag{15}$$

$$I_I(k,m,n) = -ka\int_k^{\infty} \frac{J_0^2(a\mu)a\mu}{(\alpha_m^2 - a^2\mu^2)(\alpha_n^2 - a^2\mu^2)\sqrt{\mu^2 - k^2}} d\mu. \tag{16}$$

It can be seen that the numerators and denominators of these integrals are simultaneously zero when $\alpha_m = a\mu$ or $\alpha_n = a\mu$, so these are indeterminate points. Using Taylor's series, it can be shown that the functions are actually continuous. Hence, the integrals can be solved numerically so long as these indeterminate points are avoided. Also, the weak singularity at $\mu = k$ can be removed by means of suitable substitutions, as shown in the following sections. However, the integrands are strongly oscillatory and the integral I_I has an infinite limit. Therefore, it is more efficient to solve these integrals analytically to yield fast converging expansions.

B. Solution of the finite integral

After substituting $\mu = k\sqrt{1-t^2}$, Eq. (15) for the finite integral becomes

$$I_F(k,m,n) = \int_0^1 \frac{k^2a^2 J_0^2(ka\sqrt{1-t^2})}{(\alpha_m^2 - k^2a^2(1-t^2))(\alpha_n^2 - k^2a^2(1-t^2))} dt. \tag{17}$$

The Bessel functions can then be expanded using the following Lommel expansion:[6]

$$J_0(ka\sqrt{1-t^2}) = \sum_{p=0}^{\infty} \frac{J_p(ka)}{p!}\left(\frac{ka}{2}\right)^p t^{2p}, \tag{18}$$

so that the expanded integral can be written

$$I_F(k,m,n) = k^2a^2 \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{J_p(ka)J_q(ka)}{p!\,q!}\left(\frac{ka}{2}\right)^{p+q} \times \int_0^1 \frac{t^{2p+2q}}{(\alpha_m^2 - k^2a^2(1-t^2))(\alpha_n^2 - k^2a^2(1-t^2))} dt. \tag{19}$$

After solving the integral,[3] and truncating the infinite expansion limits to P and Q, the final expression can be written

$$I_F(k,m,n)|_{m\neq n} = k^2a^2 \sum_{p=0}^{P}\sum_{q=0}^{Q} \frac{J_p(ka)J_q(ka)}{p!\,q!\,(2p+2q+1)(\alpha_m^2 - \alpha_n^2)} \times \left(\frac{ka}{2}\right)^{p+q}\left\{\frac{1}{k^2a^2 - \alpha_m^2}\,{}_2F_1\left(1, u-1; u; \frac{k^2a^2}{k^2a^2 - \alpha_m^2}\right) - \frac{1}{k^2a^2 - \alpha_n^2}\,{}_2F_1\left(1, u-1; u; \frac{k^2a^2}{k^2a^2 - \alpha_n^2}\right)\right\}, \tag{20}$$

where $u = p + q + 3/2$ and which is valid for $m \neq n$. It can be seen that if $m = n$, then $\alpha_m^2 - \alpha_n^2 = 0$, which leads to a singular expression. In the case of the shell, this problem did not arise due to the presence of damping resistance in the boundary conditions, which gave rise to complex eigenvalues so that the difference between the squares of two complex conjugate eigenvalues was non-zero. Here, for $m = n$ the following alternative expression has to be used:

$$I_F(k,m)|_{m=n} = k^2a^2 \sum_{p=0}^{P}\sum_{q=0}^{Q} \frac{J_p(ka)J_q(ka)}{p!\,q!\,(2p+2q+1)(k^2a^2 - \alpha_m^2)^2} \times \left(\frac{ka}{2}\right)^{p+q} {}_2F_1\left(2, u-1; u; \frac{k^2a^2}{k^2a^2 - \alpha_m^2}\right). \tag{21}$$

C. Solution of the infinite integral

After substituting $\mu = k\sqrt{t^2+1}$, Eq. (16) for the finite integral becomes

$$I_I(k,m,n) = -\int_0^\infty \frac{k^2a^2J_0^2(ka\sqrt{t^2+1})}{(\alpha_m^2 - k^2a^2(t^2+1))(\alpha_n^2 - k^2a^2(t^2+1))}dt. \tag{22}$$

The Bessel functions can then be expanded using Gegenbauer's summation theorem[5] as follows:

$$J_0(ka\sqrt{t^2+1}) = 2\sum_{p=0}^{\infty} \frac{(-1)^p}{1+\delta_{p0}} J_{2p}(ka)J_{2p}(kat), \tag{23}$$

so that the expanded integral can be written

$$I_I(k,m,n) = -4k^2a^2\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{(-1)^{p+q}J_{2p}(ka)J_{2q}(ka)}{(1+\delta_{p0})(1+\delta_{q0})} \times \int_0^\infty \frac{J_{2p}(kat)J_{2q}(kat)}{(\alpha_m^2 - k^2a^2(t^2+1))(\alpha_n^2 - k^2a^2(t^2+1))}dt. \tag{24}$$

After solving the integral,[3] and truncating the infinite expansion limits to P and Q, the final expression can be written

$$I_I(k,m,n)|_{m\neq n} = -\frac{ka}{\pi}\sum_{p=0}^{P}\sum_{q=0}^{Q} \frac{J_{2p}(ka)J_{2q}(ka)}{(1+\delta_{p0})(1+\delta_{q0})(v-2)_2(u-2)_2(\alpha_m^2-\alpha_n^2)} \times \left\{ {}_3F_4\left(1,1,\frac{3}{2};3-u,u,3-v,v;k^2a^2-\alpha_m^2\right) - {}_3F_4\left(1,1,\frac{3}{2};3-u,u,3-v,v;k^2a^2-\alpha_n^2\right)\right\}, \tag{25}$$

where $u=p+q+3/2$, $v=p-q+3/2$ and which is valid for $m\neq n$. Again, in order to avoid singularities, for $m=n$ the following alternative expression is provided:

$$I_I(k,m)|_{m=n} = \frac{3ka}{2\pi}\sum_{p=0}^{P}\sum_{q=0}^{Q} \frac{J_{2p}(ka)J_{2q}(ka)}{(1+\delta_{p0})(1+\delta_{q0})(u-3)_4(v-3)_4} \times {}_3F_4\left(2,2,\frac{5}{2};4-u,u+1,4-v,v+1;k^2a^2-\alpha_m^2\right). \tag{26}$$

It should be noted that in the case of the shell,[2] the solution to the infinite integral also contained Bessel functions of the second kind, in addition to the hypergeometric functions, due to the eigenvalues being complex. Here they are real, so those Bessel functions would not be valid and have been removed in order to obtain the correct result.

VII. FAR FIELD PRESSURE RESPONSE

The far field pressure is derived by inserting the following far-field Green's function in spherical-cylindrical coordinates

$$g(r,\theta,\phi|w_0,\phi_0,z_0) = \frac{1}{4\pi r}e^{-ik(r-w_0\sin\theta\cos(\phi-\phi_0)-z_0\cos\theta)} \tag{27}$$

in Eq. (3) and integrating over the surface of the membrane to obtain

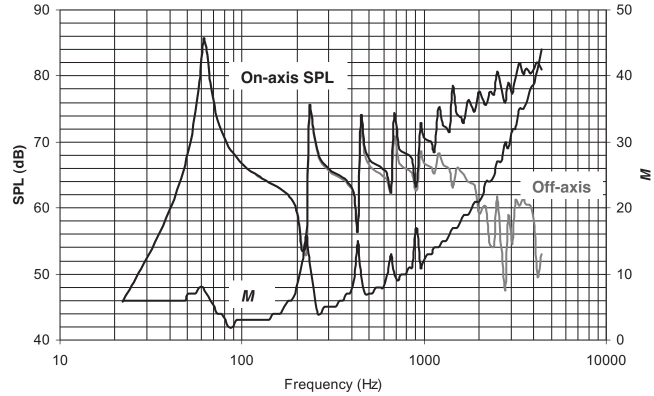

FIG. 2. On-axis and 45° off-axis pressure responses and summation limit M using the spheroidal trial function of Eq. (1), where $a=125$ mm, $h=12$ μm, $T=100$ N/m, $\rho_D=1667$ kg/m^3, $p_I=1$ Pa, and $r=1$ m.

$$\tilde{p}(r,\theta) = -\frac{ia}{4r}\tilde{p}_I e^{-ikr}D(\theta), \tag{28}$$

where r is the distance from the origin to the observation point and θ is the angle subtended to the axis of symmetry. The directivity function $D(\theta)$ is given by

$$D(\theta) = 2ka\sum_{m=1}^{M} \frac{\alpha_m J_1(\alpha_m)J_0(ka\sin\theta)}{\alpha_m^2 - (ka\sin\theta)^2}\tau_m. \tag{29}$$

For $\theta=0$ (i.e., on axis), this simplifies to

$$D(0) = 2ka\sum_{m=1}^{M} \frac{J_1(\alpha_m)}{\alpha_m}\tau_m. \tag{30}$$

The on-axis pressure response without damping is shown in Fig. 3 where the sound pressure level (SPL) is given by: $SPL=20\log_{10}|\tilde{p}(r,0)/(20\times 10^{-6})|$. Figure 2 shows the response calculated using the surface-velocity trial function given in Eq. (1) and the far-field expression given in the previous paper.[1] The summation limit M is shown for each of the 114 points plotted. The summation limit was determined using a convergence criterion of 0.002 dB maximum difference between the SPL computed with M and $(M+1)$ terms. The remaining summation limits P and Q were set to $P=Q=2M$.

VIII. DISCUSSION OF THE RESULTS

The on-axis pressure responses shown in Figs. 2 and 3 are fairly typical of clamped membranes and plates in general. At the eigenfrequencies the membrane's displacement becomes very large. When most of the membrane's surface is moving in the same direction, large peaks are produced. Also, there are frequencies at which the displacement is still large, but different regions of the membrane are moving in opposite directions so as to cancel each other out: Instead of radiating sound, air is just moved back and forth between the areas of the surface that are moving in opposite phase. An excellent illustration of this has been provided by Streng.[7] The damping effect of the increasingly resistive radiation impedance at high frequencies is fairly evident. The direct nature of calculation method shows considerable detail in this region, enabled by the elimination of collocation or least

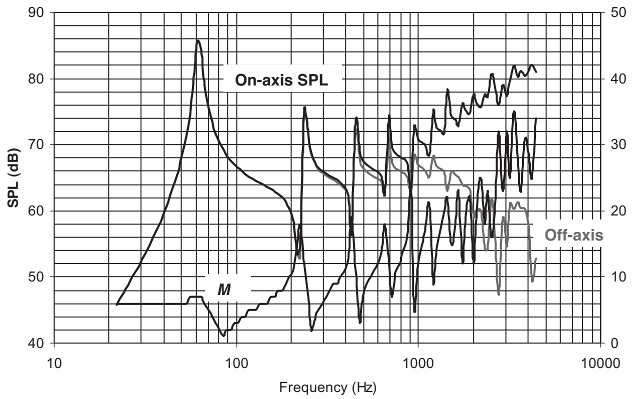

FIG. 3. On-axis and 45° off-axis pressure responses and summation limit M using the eigenfunction-expansion trial function of Eq. (2) with same parameters as Fig. 2.

squares methods. The actual differences between the pressures in Figs. 2 and 3 are fairly small. On axis, the maximum difference is 0.03 dB, rising to 0.1 dB at 45° off axis. As expected, the off axis response rolls off early due to high-frequency beaming, but still shows significant modal variations.

Also shown in these figures are the numbers of terms in the summations required to meet the convergence criterion of less than 0.002 dB difference in SPL when one extra term is added to the summation. Not surprisingly, it can be seen that generally more terms are required at higher frequencies due to the increasing complexity in the displaced membrane shape. Also, more terms are required at the eigenfrequencies because the driving force and acoustic pressure are distributed in such a way that excites not just that particular eigenfrequency, but a complex mixture of higher order modes too. Interestingly, fewer terms are required in the case of the eigenfunction method in between the eigenfrequencies, so here it must provide a more natural fit to the displaced shape. However, at the eigenfrequencies, the summation limit tends to be slightly greater than with the spheroidal method.

Regarding the speed of the calculations, this would of course vary considerably depending on whether a compiled or non-compiled program is used, numerical algorithms, type of hardware and other factors too. Suffice to say that the spheroidal method was found to be about ten times faster. This is mainly due to the fact that the hyperbolic functions in Eqs. (21) and (26) are functions of both p and q in the double summations, whereas in the case of the spheroidal method[1] the hypergeometric function is a function of q only.

IX. CONCLUSIONS

An alternative method has been presented for calculating the radiation characteristics of a membrane using a trial function based upon an eigenfunction expansion. This method, like the previous one which used a spheroidal trial function, does not require numerical integration.

Since the two methods give SPL values within 0.03 dB of each other on axis, it is not possible to state that one is necessarily more accurate than the other, but the fact that the two results are so consistent means that neither is likely to be wildly inaccurate.

The simpler formulas provided by the spheroidal series make it easier to derive expressions for other radiation characteristics such as the near field pressure and radiation impedance. Unfortunately, it appears to be restricted to cases where the surface velocity/deflection is either zero[1] or infinity[8] at the rim, such as circular membranes,[1] sound holes,[8] and clamped or simply supported plates and shells. Otherwise, for a finite rim velocity, the eigenfunction expansion has to be used. Examples of the latter are circular plates or shells[2] with non-zero rim velocity as in dynamic loudspeakers, earpieces and microphones, where the contribution of the surround is ignored (but its stiffness can still be included in the boundary conditions[2]). The rim velocity boundary condition affects the modal structure which, in turn, affects the computed sound pressure. Although the eigenfunction method is slower and therefore less attractive for the current problem, it has been verified and can therefore be applied with confidence to other uses where the spheroidal method is unsuitable.

[1]T. J. Mellow and L. M. Kärkkäinen, "On the sound field of a circular membrane in free space and an infinite baffle," J. Acoust. Soc. Am. **120**(5), 2460–2477 (2006).

[2]T. J. Mellow and L. M. Kärkkäinen, "On the sound field of a shallow spherical shell in an infinite baffle," J. Acoust. Soc. Am. **121**(6), 3527–3541 (2007).

[3]S. Wolfram, *The Mathematica Book*, 5th ed. (Wolfram Media, Champaign, IL, 2003).

[4]P. M. Morse and K. U. Ingard, *Theoretical Acoustics* (McGraw–Hill, New York, 1968), pp. 320, 365, 389, 390.

[5]I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 6th ed., edited by A. Jeffrey (Academic, New York, 2000), pp. 668, Eq. (6.561.5), pp. 930, Eq. (8.532.1), p. 980, Eqs. (8.930.1)–(8.930.7).

[6]G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd ed. (Cambridge University Press, London, 1944), p. 141, Sec. 5.22, Eq. (5).

[7]J. H. Streng, "Calculation of the surface pressure on a vibrating circular stretched membrane in free space," J. Acoust. Soc. Am. **82**(2), 679–686 (1987).

[8]T. J. Mellow, "On the sound field of a resilient disk in an infinite baffle," J. Acoust. Soc. Am. **120**(1), 90–101 (2006).